[11-05-17-T8] REV1

Square roots

DEFINITION. If $b = a \cdot a$, then a is a **square root** of b.

THEOREM. Every positive number has two square roots. The square root of zero is zero. The square root of a negative number is not a real number.

The symbol $\sqrt{a}$ denotes to positive square root of a and $-\sqrt{a}$ denotes the negative square root of a. For example, the positive square root of 9 is $\sqrt{9}$ which is 3, and the negative square root of 9 is $-\sqrt{9}$ which is -3.

THEOREM. $\sqrt{a^2} = a$.

THEOREM. For positive numbers a and b, if $a < b$ then $\sqrt{a} < \sqrt{b}$.

THEOREM. Products and quotients of square roots. $\sqrt{a}\,\sqrt{b} = \sqrt{a\,b}$ and $\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$, for a and b positive integers.

The definition and theorems above are all the facts you need to know, for the time being, to work with square roots. Once you understand these facts, it is a matter of practice in their application. You cannot possibly anticipate every combination of numbers and square roots that might arise in your work in mathematics. Therefore, you must know the above facts and become skilled at applying them in particular cases.

Rationalizing the denominator. We usually prefer that square roots do not appear in the denominator of rational expressions. It is an easy matter to rewrite such an expression with a denominator that is not a square root. The following procedure will always accomplish this task.

$$\frac{a}{\sqrt{b}} = \frac{a}{\sqrt{b}} \cdot \frac{\sqrt{b}}{\sqrt{b}} = \frac{a\sqrt{b}}{b}.$$

Typical computations that involve square roots.

[1] Simplify $\sqrt{36}$. Since $36 = 6^2$, $\sqrt{36} = 6$.

[2] Simplify $\sqrt{27}$. Solution. $\sqrt{27} = \sqrt{3^2 \cdot 3} = \sqrt{3^2} \cdot \sqrt{3} = 3\sqrt{3}$.

[3] Simplify $\sqrt{32}$. Solution. $\sqrt{32} = \sqrt{4^2 \cdot 2} = \sqrt{4^2} \cdot \sqrt{2} = 4\sqrt{2}$.

[4] Simplify $\sqrt{2^4 \cdot 5^2 \cdot 7}$.

Solution. $\sqrt{2^4 \cdot 5^2 \cdot 7} = \sqrt{2^2 \cdot 2^2 \cdot 5^2 \cdot 7} = \sqrt{2^2} \cdot \sqrt{2^2} \cdot \sqrt{5^2} \cdot \sqrt{7} = 2 \cdot 2 \cdot 5\sqrt{7} = 20\sqrt{7}$.

[5] Simplify $\sqrt{3528}$. Solution. $\sqrt{3528} = \sqrt{2^3\, 3^2\, 7^2} = \sqrt{2^2 \cdot 2 \cdot 3^2 \cdot 7^2} = 2 \cdot 3 \cdot 7\sqrt{2} = 42\sqrt{2}$.

Note that one may always write a number that is non-prime as a product of prime factors. Discovering that product can be done with "French division". For example,

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3528 =
 2 | 1764
 2 | 882
 2 | 441
 3 | 147
 3 | 49
 7 | 49
 7 | 7
   | 1
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So, $\sqrt{3528} = \sqrt{2^3\, 3^2\, 7^2}$

No law says you must rewrite a number as a product of primes before taking its square root. If you happen to see that $\sqrt{1800} = \sqrt{9 \cdot 100 \cdot 2}$, then by all means conclude that $\sqrt{1800} = 3 \cdot 10 \sqrt{2} = 30 \sqrt{2}$. Or, perhaps you notice that $\sqrt{1800} = \sqrt{900 \cdot 2} = \sqrt{30^2 \cdot 2}$, so you write $\sqrt{1800} = 30 \sqrt{2}$.

[6] Simplify $\sqrt{\frac{9}{25}}$. Solution. $\sqrt{\frac{9}{25}} = \frac{\sqrt{9}}{\sqrt{25}} = \frac{3}{5}$.

[7] Simplify $\sqrt{\frac{9}{8}}$. Solution. $\sqrt{\frac{9}{8}} = \frac{\sqrt{9}}{\sqrt{8}} = \frac{3}{2\sqrt{2}} = \frac{3}{2\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{3\sqrt{2}}{4}$. Here we rationalized the denominator.

[8] Simplify $\sqrt{3} \cdot \sqrt{5}$. Solution. $\sqrt{3}\ \sqrt{5} = \sqrt{15}$.

[9] Simplify $\sqrt{18} \cdot \sqrt{75}$. Solution. $3\sqrt{2} \cdot 5\sqrt{3} = 10\sqrt{6}$.

[10] Simplify $\sqrt{12} \cdot \sqrt{45}$. Solution. $2\sqrt{3} \cdot 3\sqrt{5} = 6\sqrt{15}$.

[11] Simplify $\sqrt{16} \cdot \sqrt{144}$. Solution. $4 \cdot 12 = 48$.

Note that in examples [9], [10], and [11], we simplified each square root before computing the product. The alternative - multiply first, then simplify - is perfectly legal, but usually poor strategy due to the effort involved. For example, working example [10] this way leads to $\sqrt{12} \cdot \sqrt{45} = \sqrt{12 \cdot 45} = \sqrt{540} = \sqrt{2^2 \cdot 3^3 \cdot 5} = \sqrt{2^2\, 3^2\, 3 \cdot 5} = 2 \cdot 3\sqrt{3 \cdot 5} = 6\sqrt{15}$. Do you really want to do all that?

[12] Simplify $\sqrt{\frac{32}{75}}$. Solution. $\sqrt{\frac{32}{75}} = \frac{\sqrt{32}}{\sqrt{75}} = \frac{4\sqrt{2}}{5\sqrt{3}} = \frac{4\sqrt{6}}{15}$. Can you see that we rationalized the denominator by multiplying the second to last expression by $\frac{\sqrt{3}}{\sqrt{3}}$?

[13] Simplify $\sqrt{20} + \sqrt{27}$. Solution. $2\sqrt{5} + 3\sqrt{3}$. That's it, because these are "unlike" terms in that the terms are not multiples of the same square root.

[14] Simplify $\sqrt{20} + \sqrt{90}$. Solution. $2\sqrt{5} + 3\sqrt{5} = 5\sqrt{5}$. Two of some thing and 3 of the same thing makes 5 of them. Here the "thing" happens to be the quare root of five.

[15] If $\sqrt{7} \approx 2.65$, then what is $\sqrt{700}$? Solution. $\sqrt{700} = \sqrt{100 \cdot 7} = 10\sqrt{7} \approx 10 \cdot 2.65 = 26.5$.

[16] If $\sqrt{.6} \approx 2.45$, then what is $\sqrt{.06}$? Solution. $\sqrt{.06} = \sqrt{\frac{6}{100}} = \frac{1}{10}\sqrt{6} \approx \frac{1}{10} \cdot 2.45 = 0.245$.

[17] Simplify $\frac{3\sqrt{5}-2\sqrt{7}}{3}$. Don't fall to temptation! There is no cancellation here.

[18] Simplify $\frac{3\sqrt{5}-6\sqrt{7}}{3} = \frac{3\left(\sqrt{5}-2\sqrt{7}\right)}{3} = \sqrt{5} - 2\sqrt{7}$.

[19] Simplify $\sqrt{\sqrt{80} \cdot \sqrt{250}}$. Solution. $\sqrt{4\sqrt{10} \cdot 5\sqrt{10}} = \sqrt{20 \cdot 10} = \sqrt{2^2 \cdot 5 \cdot 2 \cdot 5} = 10\sqrt{5}$

[20] Simplify $\frac{\sqrt{2}\left(5\sqrt{18}+3\sqrt{50}\right)}{\sqrt{360}}$.

Solution. $\frac{\sqrt{2}\left(5\sqrt{18}+3\sqrt{50}\right)}{\sqrt{360}}$

$$= \frac{\sqrt{2}\left(15\sqrt{2}+15\sqrt{2}\right)}{\sqrt{36 \cdot 10}}$$
$$= \frac{\sqrt{2}\left(30\sqrt{2}\right)}{6\sqrt{10}}$$
$$= \frac{60}{6\sqrt{10}}$$
$$= \frac{10}{\sqrt{10}}$$
$$= \frac{10\sqrt{10}}{10}$$
$$= \sqrt{10}$$

The individual steps were not hard. Overall, the solution might appear complicated. Complicated, not hard.